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Electromagnetism in non-inertial coordinates

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Abstract. The electromagnetic equations are expressed in a form appropriate to an observer in arbitrary motion in flat spacetime. The field equations preserve their form, dependence on the motion being confined to the material constitutive relations which are assumed linear and non-dispersive. The 4-metric is uniquely defined by the geometry of the observer's worldline. Two conceivable metrics for the 3-space are noticed and the choice judged unimportant for most purposes. An equation for the eikonal of geometrical optics is found and an approximate solution exhibits the leading contributions that are made to the optical length of a ray by the observer's angular velocity and translational acceleration, the former contribution being independent of the refractive index of material traversed by the ray and at rest in the observer's coordinate system.

1. Introduction

In the context of special relativity (gravity absent and consequently a flat spacetime) it is occasionally necessary to consider electromagnetism in non-inertial coordinates. An example is the Sagnac experiment and subsequent developments leading to the modern optical gyroscope. A typical situation is that an observer in arbitrary motion, and consequently using a non-inertial coordinate system, experiences electromagnetic phenomena in matter that is at rest in his coordinate system. He seeks the field equations and constitutive relations for matter that behaves linearly and without dispersion when at rest in an inertial frame. Of course some materials behave in a much more complicated way, perhaps being non-linear or exhibiting memory effects for example, but for many purposes the simple model is useful.

Despite the limited objective, it can prove surprisingly difficult and even confusing to try to cull from the literature a succinct formulation that is sufficiently general to cover a range of conditions in a manner permitting ready application to particular cases. Several reasons may be advanced. A minor hindrance is that some of the published material is buried in accounts of general relativity although in practical terms gravity can be neglected. Some treatments are restricted to free space or time-orthogonal coordinates or to both. Another aspect concerns the metric tensor of the 3-space. Because the absence of gravity allows the existence of coordinates for which the spacetime metric is the diagonal Lorentzian $(1, 1, 1, -1)$ most accounts of special relativity tacitly assume such coordinates; in particular the 3-space metric is assumed to be that of rectangular cartesian coordinates. But in non-inertial systems there is interest also in taking for the metric of the 3-space something other than the spatial part of the metric of the 4-space. And finally, none of the usual treatments achieves useful generality regarding the kinematics of the observer's motion.

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The present purpose is to give a formulation, in terms of the kinematics of the observer's world line, that is readily applicable to a range of situations. Although attention is focused on material at rest in the observer's coordinate system, the basic relations in terms of the 4-velocity of the material are applicable to material in motion.

Section 2 establishes the notation and the point of view, setting out Maxwell's equations and the constitutive relations. Section 3 considers the metrics, both of the 4-space and the 3-space, and the consequences of various choices. Section 4 gives some examples and makes comparisons with other treatments. In section 5 the equation for the eikonal of geometrical optics is obtained and an approximate solution provides the basis for remarks on the optical detection of angular velocity and translational acceleration.

2. The electromagnetic equations

Maxwell's equations in four-dimensional form are

$$L_{\alpha\beta,\gamma}+L_{\beta\gamma,\alpha}+L_{\gamma\alpha,\beta}=0 \tag{1a}$$

$$|G|^{-1/2}(|G|^{1/2}M^{\alpha\beta})_{,\beta}=S^{\alpha}. \tag{1b}$$

The arbitrary coordinates are x^{α} with $x^4=ct$, t being the time and c being the speed of light in an inertial frame. A comma denotes partial differentiation with respect to the coordinate corresponding to the following letter. Greek indices run from 1 to 4 and Latin indices run from 1 to 3, repeated indices implying summation. The metric 4-tensor $g_{\alpha\beta}$ has determinant $G<0$ and $g_{44}<0$. $L_{\alpha\beta}$ and $M^{\alpha\beta}$ are antisymmetric 4-tensors associated, respectively, with the field 3-vector pairs $(\boldsymbol{E}, \boldsymbol{B})$ and $(\boldsymbol{H}, \boldsymbol{D})$. The right-hand side of (1b) takes account of the charges and currents that are the sources of the field.

We identify the components of the field 3-vectors in terms of the elements of the antisymmetric 4-tensors in such a way that Maxwell's equations in 3-space take their customary form

$$\varepsilon^{mnp}E_{p,n}=-\partial B^m/\partial t \qquad (\Gamma^{1/2}B^m)_{,m}=0 \tag{2a}$$

$$\varepsilon^{mnp}H_{p,n}=J^m+\partial D^m/\partial t \qquad \Gamma^{-1/2}(\Gamma^{1/2}D^m)_{,m}=\rho \tag{2b}$$

the charge density and current density being ρ and J^m, respectively. With $e^{mnp}=e_{mnp}=\pm 1$ according as the indices are an even or odd permutation of natural order,

$$\varepsilon^{mnp}=\Gamma^{-1/2}e^{mnp} \qquad \varepsilon_{mnp}=\Gamma^{1/2}e_{mnp}$$

the metric of the 3-space being γ_{mn} with determinant $\Gamma>0$. Indices in (2) are raised and lowered by this metric 3-tensor which, for reasons discussed later, may differ from the spatial part of the metric 4-tensor.

With the restriction that Γ is independent of time, it is straightforward to show that (1a) implies (2a) and (1b) implies (2b) subject to the identifications

$$L_{mn}=c\varepsilon_{mnp}B^p \qquad L_{n4}=E_n \tag{3}$$

$$M^{mn}=\Lambda\varepsilon^{mnp}H_p \qquad M^{m4}=-\Lambda cD^m \tag{4}$$

$$S^m=\Lambda J^m \qquad S^4=\Lambda c\rho \qquad \Lambda=(\Gamma|G|^{-1})^{1/2}. \tag{5}$$

The constitutive relations are expressed in terms of the 4-tensors dual to $L_{\alpha\beta}$ and $M^{\alpha\beta}$. With $e^{\alpha\beta\gamma\delta} = e_{\alpha\beta\gamma\delta} = \pm 1$ according as the indices form an even or odd permutation of natural order, define

$$\varepsilon_{\alpha\beta\gamma\delta} = |G|^{1/2} e_{\alpha\beta\gamma\delta} \qquad \varepsilon^{\alpha\beta\gamma\delta} = -|G|^{-1/2} e^{\alpha\beta\gamma\delta}$$

$$P^{\alpha\beta} = \tfrac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} L_{\gamma\delta} \qquad P^{mn} = -\Lambda \varepsilon^{mnp} E_p \qquad P^{m4} = -\Lambda c B^m$$

$$Q_{\alpha\beta} = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} M^{\gamma\delta} \qquad Q_{mn} = -c\varepsilon_{mnp} D^p \qquad Q_{m4} = H_m.$$

At this point attention is for the moment confined to inertial coordinates. In an inertial frame in which the material is at rest and unaccelerated we take the constitutive relations to be

$$B^m = \mu_0 \mu^{mn} H_n \qquad D^m = \varepsilon_0 \varepsilon^{mn} E_n. \tag{6}$$

SI units are used throughout, μ_0 and ε_0 being, respectively, the permeability and permittivity of free space. Define the 4-tensor of relative permeability $\mu^{\alpha\beta}$ by its components in the rest inertial system, noting that the 4-velocity of the material is $\nu^\alpha = c\delta_4^\alpha$ in that system. Hence in any inertial system we have the Minkowski magnetic constitutive relation

$$\nu_\beta P^{\alpha\beta} = (\mu_0/\varepsilon_0)^{1/2} \mu^{\alpha\beta} \nu^\lambda Q_{\beta\lambda}. \tag{7}$$

Similarly the electric constitutive relation is

$$\nu_\beta M^{\alpha\beta} = (\varepsilon_0/\mu_0)^{1/2} \varepsilon^{\alpha\beta} \nu^\lambda L_{\beta\lambda}. \tag{8}$$

If the material is isotropic in the inertial rest frame and has relative permeability μ then

$$\mu^{\alpha\beta} = \mu(g^{\alpha\beta} + c^{-2}\nu^\alpha \nu^\beta)$$

and $\varepsilon^{\alpha\beta}$ takes a similar form. The constitutive relations become

$$\nu_\beta P^{\alpha\beta} = (\mu_0/\varepsilon_0)^{1/2} \mu \nu^\lambda g^{\alpha\beta} Q_{\beta\lambda} \tag{9}$$

$$\nu_\beta M^{\alpha\beta} = (\varepsilon_0/\mu_0)^{1/2} \varepsilon \nu^\lambda g^{\alpha\beta} L_{\beta\lambda}. \tag{10}$$

We now assume that (7)-(10) hold also in any non-inertial coordinate system. Thus we assume that, in the momentarily comoving inertial frame, the constitutive relations are unaffected by the acceleration. In the absence of a supporting microscopic model this is a mere assumption which can, however, be defended on two grounds. Firstly the assumption is likely to be approximately true of all materials at low accelerations whereas any other behaviour is likely to be restricted to specific classes of material. Secondly the model implied by the assumption provides a benchmark against which other models may be viewed.

In what follows we consider the electric and magnetic constitutive relations in the simple illustrative case where the material is at rest in the observer's non-inertial frame, and would be isotropic if it were at rest in an inertial frame. Thus the constitutive relations are (9) and (10) with

$$\nu^\alpha = c|g_{44}|^{-1/2}\delta_4^\alpha \qquad \nu_\alpha = c|g_{44}|^{-1/2} g_{\alpha 4}.$$

The results are

$$D^m = \varepsilon\varepsilon_0 \Psi^m_n E^n - \varepsilon^{mnp} g_n H_p \qquad B^m = \mu\mu_0 \Psi^m_n H^n + \varepsilon^{mnp} g_n E_p \tag{11}$$

$$\Psi^m_n = |\Lambda g_{44}|^{-1} g^{mp} \gamma_{pn} \qquad g_n = |c g_{44}|^{-1} g_{n4}. \tag{12}$$

The set of 3-vectors, comprising g_n together with the field vectors, have their indices raised and lowered by the metric 3-tensor γ_{mn}. It may be remarked that the relation

$$\mu\mu_0 H_m D^m = \varepsilon\varepsilon_0 E_m B^m$$

which is trivially obvious in inertial coordinates, carries over to non-inertial coordinates.

Finally the 4-force on a particle of charge q is

$$P_\alpha = (q/c)\nu^\beta L_{\alpha\beta} \tag{13}$$

from which we can find the 3-force in the frame of the non-inertial observer once we have discussed the metrics.

3. The metrics and their consequences

In time-orthogonal coordinates, i.e. with $g_{m4} = 0$, the metric of the 3-space, γ_{mn}, is g_{mn}, the spatial part of the metric of the 4-space. One reason why we have preserved the distinction between them becomes clear from (12), which suggests that freedom to choose γ_{mn} independently of g_{mn} may enable us to simplify the consitutive relations. In particular, if we could have $g^{mp}\gamma_{pn} = \delta^m_n$ then the tensor Ψ^m_n would be isotropic. But clearly this condition is not in general satisfied by choosing $\gamma_{mn} = g_{mn}$. In fact

$$g^{mp}g_{pn} = \delta^m_n - g^{m4}g_{4n}.$$

But there is another reason for maintaining the distinction between the two 3-metrics. As remarked in section 1, some of the work that is relevant is buried in accounts of general relativity in those parts (and this is the crucial point) where the curvature of spacetime does not feature. In other words the relevant discussion is independent of whether spacetime is flat or curved. In this context both Møller (1952, pp 237-8) and Landau and Lifshitz (1962, pp 271-3) consider the concepts of proper time and proper distance. On putting $\mathrm{d}x^m = 0$ in

$$-c^2\,\mathrm{d}\tau^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$

the element of proper time is

$$\mathrm{d}\tau = c^{-1}|g_{44}|^{1/2}\,\mathrm{d}x^4.$$

But the element of proper distance is

$$\mathrm{d}\sigma \neq (g_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n)^{1/2}.$$

By proper distance between adjacent events is meant the distance measured by a momentarily comoving inertial observer, and the two cited references show that

$$\mathrm{d}\sigma = (\gamma_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n)^{1/2} \qquad \gamma_{mn} = g_{mn} - g_{44}^{-1}g_{m4}g_{n4}. \tag{14}$$

This choice has two consequences. Firstly on expanding the determinant of the 4-metric by pivotal condensation on g_{44} it is seen that $G = g_{44}\Gamma$. Secondly from

$$\delta^\alpha_\beta = g^{\alpha p}g_{p\beta} + g^{\alpha 4}g_{4\beta}$$

$$0 = \delta^n_4 = g^{np}g_{p4} + g^{n4}g_{44}$$

$$g^{n4} = -g_{44}^{-1}g^{np}g_{p4}.$$

Substituting the left-hand side of the last equation into the expansion of $g^{np}\gamma_{pm}$ gives $g^{np}\gamma_{pm}=\delta^n_m$, as desired.

Thus the choice (14) gives

$$\begin{aligned} &\Lambda=|g_{44}|^{-1/2} \qquad \Psi^m_n=|g_{44}|^{-1/2}\delta^m_n \\ &D^m=|g_{44}|^{-1/2}\varepsilon\varepsilon_0 E^m-\varepsilon^{mnp}g_n H_p \\ &B^m=|g_{44}|^{-1/2}\mu\mu_0 H^m+\varepsilon^{mnp}g_n E_p. \end{aligned} \tag{15}$$

The case of time-orthogonal coordinates is particularly simple, for then g_n vanishes and the sole effect of the non-inertial coordinates is to multiply the relative permittivity and relative permeability by the factor $|g_{44}|^{-1/2}$. This is true even for free space.

The form of the metric 4-tensor has now to be considered. The details are relegated to the appendix and an outline will suffice here. An observer in arbitrary motion may use a wide range of coordinate systems, but one merits particular attention because it is completely defined by the geometry of his worldline in analogy to the Frenet triad of orthonormal vectors encountered in the geometry of a curve in three dimensions. In that case, beginning with unit tangent, repeated differentiation generates unit normal and unit binormal, introducing the scalar curvature and torsion. In an analogous fashion, beginning with $\boldsymbol{X}_4$ the timelike unit tangent to the observer's world line, repeated differentiation generates the orthonormal tetrad $\boldsymbol{X}_\alpha$ that is the vector basis for the observer's coordinate system. The spacelike triad $\boldsymbol{X}_m$ is the vector basis for the 3-space, and an event P in it has position vector $\boldsymbol{r}=x^m\boldsymbol{X}_m$. If T is the observer's proper time the spacetime coordinates that he allocates to P are x^α with $x^4=cT$.

The Frenet equations for the observer's worldline take the form

$$\begin{aligned} &\mathrm{d}\boldsymbol{X}_4/\mathrm{d}T=(f/c)\boldsymbol{X}_1 \qquad \mathrm{d}\boldsymbol{X}_1/\mathrm{d}T=\boldsymbol{\Omega}\times\boldsymbol{X}_1+(f/c)\boldsymbol{X}_4 \\ &\mathrm{d}\boldsymbol{X}_2/\mathrm{d}T=\boldsymbol{\Omega}\times\boldsymbol{X}_2 \qquad \mathrm{d}\boldsymbol{X}_3/\mathrm{d}T=\boldsymbol{\Omega}\times\boldsymbol{X}_3. \end{aligned}$$

The first, second and third unit normals are, respectively, $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$, and the first, second and third curvatures are, respectively, K_1, K_2 and K_3. The physical significance of the first curvature is that the observer's proper acceleration is

$$\boldsymbol{f}=f\boldsymbol{X}_1 \qquad f=c^2K_1.$$

The physical significance of the second and third curvatures arises from the definition

$$\boldsymbol{\Omega}=c(K_2\boldsymbol{X}_3+K_3\boldsymbol{X}_1)$$

for the Frenet equations show that $\boldsymbol{\Omega}$ is the angular velocity of $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$ and also gives the spatial part of $\mathrm{d}\boldsymbol{X}_1/\mathrm{d}T$. Here and subsequently, when working with 3-vectors, it is convenient to use the customary notation in which the dot and cross, respectively, denote scalar and vector products.

This coordinate system may fairly be called the observer's intrinsic system, and in it the square of the element of interval is

$$\begin{aligned} &\mathrm{d}s^2=\partial\boldsymbol{r}\cdot\partial\boldsymbol{r}+2\boldsymbol{h}\cdot\partial\boldsymbol{r}\,\mathrm{d}x^4-[(c_1/c)^2-h^2](\mathrm{d}x^4)^2 \\ &\boldsymbol{h}=\boldsymbol{\Omega}\times\boldsymbol{r}/c \qquad c_1/c=1+\boldsymbol{r}\cdot\boldsymbol{f}/c^2. \end{aligned} \tag{16}$$

In this notation $\partial\boldsymbol{r}=\mathrm{d}x^m\boldsymbol{X}_m$. Hence

$$\begin{aligned} &g_{mn}=\delta_{mn} \qquad g_{44}=-[(c_1/c)^2-h^2] \\ &\boldsymbol{g}=|cg_{44}|^{-1}\boldsymbol{h} \qquad G=-(c_1/c)^2. \end{aligned}$$

In dyadic notation, U being the unit dyadic,

$$g^{mn} = \boldsymbol{X}_m \cdot [\mathsf{U} - (c/c_1)^2 \boldsymbol{hh}] \cdot \boldsymbol{X}_n.$$

This coordinate system is not time-orthogonal in general, and choosing the 3-metric according to (14) leads as before to the constitutive relations (15) and

$$\Gamma = (c_1/c)^2[(c_1/c)^2 - h^2]^{-1} \qquad \Lambda = [(c_1/c)^2 - h^2]^{-1/2}.$$

On the other hand, the choice $\gamma_{mn} = g_{mn}$ is very natural, and the dyadic representation of Ψ^m_n is seen to be

$$\boldsymbol{\Psi} = (c_1/c)[(c_1/c)^2 - h^2]^{-1}[\mathsf{U} - (c/c_1)^2 \boldsymbol{hh}]$$

giving the constitutive relations

$$\begin{aligned} [(c_1/c)^2 - h^2]\boldsymbol{D} &= \varepsilon\varepsilon_0 (c_1/c)[\mathsf{U} - (c/c_1)^2 \boldsymbol{hh}] \cdot \boldsymbol{E} - \boldsymbol{h} \times \boldsymbol{H}/c \\ [(c_1/c)^2 - h^2]\boldsymbol{B} &= \mu\mu_0 (c_1/c)[\mathsf{U} - (c/c_1)^2 \boldsymbol{hh}] \cdot \boldsymbol{H} + \boldsymbol{h} \times \boldsymbol{E}/c. \end{aligned} \tag{17}$$

Generally h is very small, and neglecting h^2 gives

$$\boldsymbol{D} = (c/c_1)(\varepsilon\varepsilon_0 \boldsymbol{E} - \boldsymbol{h} \times \boldsymbol{H}/c_1) \qquad \boldsymbol{B} = (c/c_1)(\mu\mu_0 \boldsymbol{H} + \boldsymbol{h} \times \boldsymbol{E}/c_1) \tag{18}$$

which also results from (15) with the same approximation.

We can now return to (13), giving the 4-force on a charged particle, and deduce the 3-force in the observer's coordinate system. The definition of 3-force in a non-inertial coordinate system has been discussed previously (Scorgie 1990) and it has been shown that a useful definition, as in an inertial frame, is $F_m = \beta^{-1} P_m$ with $\beta = \mathrm{d}T/\mathrm{d}\tau$, the proper time of the particle being τ. Writing

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r}$$

the 4-velocity of the particle, from (A.2), is

$$\boldsymbol{V} = \beta(\boldsymbol{v} + c_1 \boldsymbol{X}_4) = \beta(v^m \boldsymbol{X}_m + c_1 \boldsymbol{X}_4) = \nu^\alpha \boldsymbol{X}_\alpha.$$

Then (13) gives

$$P_m = (q/c)\beta(c_1 E_m + c\varepsilon_{mnp} v^n B^p).$$

Hence the 3-force is

$$\boldsymbol{F} = q[(c_1/c)\boldsymbol{E} + (\boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r}) \times \boldsymbol{B}]. \tag{19}$$

At the origin of coordinates (the location of the observer)

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \tag{20}$$

which is identical in form to the expression for the Lorentz force in an inertial frame.

A central role has been accorded to the observer's intrinsic coordinate system in which the time axis is tangent to his worldline and the space axes are its three normals; and reference should be made to other conceivable coordinate systems. The topic has previously been discussed (Scorgie 1990) and it has been shown that the metrics of coordinate systems in which the observer is at rest at the origin form a family described by the square of the element of interval

$$\mathrm{d}s^2 = \partial \boldsymbol{r} \cdot \partial \boldsymbol{r} + 2(\boldsymbol{\Omega} + \boldsymbol{\omega}) \cdot \boldsymbol{r} \times \partial \boldsymbol{r}\, \mathrm{d}T - \{c_1^2 - [(\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \boldsymbol{r}]^2\}\, \mathrm{d}T^2 \tag{21}$$

the arbitrary vector $\boldsymbol{\omega}$ being the angular velocity of the space axes as seen from the intrinsic coordinate system. The cited reference shows that the condition $\boldsymbol{\omega} = -\boldsymbol{\Omega}$ corresponds to Fermi–Walker transport of the space axes along the observer's worldline. Of course this choice abolishes rotational effects and consequently is not appropriate in a discussion of the Sagnac experiment or the optical gyroscope. The condition $\boldsymbol{\omega} = 0$ takes us back to the intrinsic coordinate system. These two cases are the most interesting for general purposes although other values of $\boldsymbol{\omega}$ may be of interest in specific instances. In any event, the structure of (21) shows that the electromagnetic constitutive relations would be obtained by replacing $\boldsymbol{\Omega}$ by $\boldsymbol{\Omega}+\boldsymbol{\omega}$ in the relations already established for the intrinsic coordinates.

4. Examples and some comparisons

The first example, from Atwater (1974), is trivial but illustrates a point that is worth remarking. He considers only free space in a frame obtained from an inertial one by the Galilean transformation

$$t = t' \qquad x = vt + x' \qquad y = y' \qquad z = z'$$

primed coordinates referring to the non-inertial frame.

In our notation

$$g_{mn} = \delta_{mn} \qquad g_{44} = -[1-(v/c)^2] \qquad g_{m4} = (v/c)\delta_{m1} \qquad G = -1.$$

Because this 4-metric is not of the form (21), we have to go back to the basic relations (11). Following Atwater, we choose $\gamma_{mn} = g_{mn}$; hence $\Lambda = 1$ and the dyadic representation of Ψ^m_n is

$$\boldsymbol{\Psi} = [1-(v/c)^2]^{-1}[\mathbf{U}-(v/c)^2\boldsymbol{ee}]$$

the unit vector in the x direction being $\boldsymbol{e}$. Also

$$\boldsymbol{g} = (v/c)^2[1-(v/c)^2]^{-1}\boldsymbol{e}.$$

The constitutive relations become

$$\begin{aligned} [1-(v/c)^2]\boldsymbol{D} &= \varepsilon\varepsilon_0[\mathbf{U}-(v/c)^2\boldsymbol{ee}]\cdot\boldsymbol{E} - (v/c^2)\boldsymbol{e}\times\boldsymbol{H} \\ [1-(v/c)^2]\boldsymbol{B} &= \mu\mu_0[\mathbf{U}-(v/c)^2\boldsymbol{ee}]\cdot\boldsymbol{H} + (v/c^2)\boldsymbol{e}\times\boldsymbol{E}. \end{aligned} \tag{22}$$

These relations hold for isotropic material, whereas Atwater deals only with free space. But even with $\varepsilon = \mu = 1$ our treatment bears little resemblance to Atwater's. To arrive at his form notice that the second of the pair (22) gives

$$[1-(v/c)^2]\boldsymbol{e}\times\boldsymbol{B} = \mu\mu_0\boldsymbol{e}\times\boldsymbol{H} + (v/c^2)\boldsymbol{e}\times(\boldsymbol{e}\times\boldsymbol{E}).$$

Substitution for $\boldsymbol{e}\times\boldsymbol{H}$ in the first member of (22), and considering only free space, gives

$$\boldsymbol{D} = \varepsilon_0(\boldsymbol{E} - v\boldsymbol{e}\times\boldsymbol{B}).$$

Then Maxwell's equation $\nabla\cdot\boldsymbol{D} = \rho$ yields

$$\nabla\cdot\boldsymbol{E} = \rho/\varepsilon_0 - v\boldsymbol{e}\cdot(\nabla\times\boldsymbol{B}) \tag{23}$$

which is one of Atwater's equations.

His approach can be summarized in this way. He defines the field in terms of only two vectors, $\boldsymbol{E}$ and $\boldsymbol{B}$, writing the inertial version of Maxwell's source equations

$$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\,\partial\boldsymbol{E}/\partial t \qquad \nabla\cdot\boldsymbol{E} = \rho/\varepsilon_0. \tag{24}$$

Since he does not recognize the vectors $\boldsymbol{D}$ and $\boldsymbol{H}$, he does not explicitly use constitutive relations of the type (22). He notes that (23) differs from the second member of (24). And a similar argument would yield an equation differing from the first member of (24). Hence Atwater concludes that the form taken by Maxwell's source equations in a non-inertial frame differs from that in an inertial frame.

Our formulation, in contrast, preserves the form of Maxwell's equations (2) in all coordinates. But the concomitant is that the field vectors to be used in the source equations are $\boldsymbol{D}$ and $\boldsymbol{H}$. If these are expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ through the constitutive relations then of course the resulting source equations do not preserve their form. The fact that we deduce Atwater's equations from our (22) shows that, for free space, both approaches lead to the same conclusions. Atwater's treatment does not extend to the case where matter is present.

An interesting example is provided by the work of Heer (1964) on the resonant frequencies of a rotating electromagnetic cavity. Two aspects present themselves: accepting Heer's metric, what form would we deduce for the constitutive equations; and, secondly, does Heer's metric accurately describe the physical situation he wishes to study?

His metric is that of an observer at rest in an inertial frame but using space axes (x, y, z) that rotate at angular velocity ω about the z axis. This is what commonly appears in the literature under the heading 'transformation to rotating axes'. It is a trivial case of (16) with the substitutions

$$T = t \qquad f = 0 \qquad \boldsymbol{\Omega} = \omega \boldsymbol{X}_3 = \boldsymbol{\omega}$$

writing $\boldsymbol{X}_3$ for the unit vector in the z direction. Consequently $c_1 = c$, $\boldsymbol{h} = \boldsymbol{\omega} \times \boldsymbol{r}/c$, and with $\gamma_{mn} = g_{mn}$, as assumed by Heer, our constitutive relation (17) is

$$(1 - h^2)\boldsymbol{D} = \varepsilon\varepsilon_0(\mathbf{U} - \boldsymbol{hh})\cdot\boldsymbol{E} - \boldsymbol{h}\times\boldsymbol{H}/c \tag{25}$$

together with a similar magnetic equation. This is an explicit relation, whereas Heer arrives at an implicit relation. To reach it, notice that the first member of (17) gives

$$\boldsymbol{D}\cdot\boldsymbol{h} = \varepsilon\varepsilon_0 \boldsymbol{E}\cdot\boldsymbol{h}$$

and this equation together with (25) gives

$$\boldsymbol{D} + \boldsymbol{h}\times(\boldsymbol{h}\times\boldsymbol{D}) = \varepsilon\varepsilon_0\boldsymbol{E} - \boldsymbol{h}\times\boldsymbol{H}/c \tag{26}$$

which is Heer's version of the electric constitutive relation. And the magnetic relation is treated similarly.

Concerning the appropriateness of the metric assumed by Heer, it is sufficient to note that he wishes to calculate the resonant frequencies of a tubular cavity in the form of a square or circle, say, rotating about an axis through its centre and perpendicular to its plane. The cavity is filled with matter that rotates with it. Consequently the matter experiences centripetal acceleration and cannot be considered at rest in an inertial frame. The worldline of a particle of matter is that of a point rotating in a circle of radius R, say, at angular velocity $\boldsymbol{\omega}$; and the Frenet relations readily give

$$K_1 = (\gamma\omega/c)^2 R \qquad K_2 = \gamma^2\omega/c \qquad K_3 = 0$$

with $\gamma = [1 - (\omega R/c)^2]^{-1/2}$.

Also the first normal to the worldline of the particle is directed towards the centre of the circle and the third normal is perpendicular to its plane. The metric is given by (16) with the substitutions

$$f = (\gamma\omega)^2 R\boldsymbol{X}_1 \qquad \boldsymbol{\Omega} = \gamma^2\omega\boldsymbol{X}_3.$$

Also $dT = \gamma^{-1}\,dt$ and the constitutive relations are (17) with

$$c_1/c = 1 + (\gamma\omega/c)^2 \boldsymbol{Rr}\cdot\boldsymbol{X}_1 \qquad \boldsymbol{h} = (\gamma^2\omega/c)\boldsymbol{X}_3\times\boldsymbol{r}.$$

Thus Heer's relation (26), involving h^2 it may be noted, is based on a metric that does not accurately describe the physical situation that he wishes to study. But the trouble is more apparent than real, for examination of his subsequent calculations shows that they retain only terms linear in h. To this order of approximation there is no difference between the two metrics.

This aspect features also in Post (1967), where he discusses at some length the correct metric to be used in treating the Sagnac effect. His preferred form results from *ad hoc* tinkering with the 'rotating axes' metric, but he concludes that the precision of experimental measurement is too low to provide a test of the various possibilities.

Møller (1952) discusses a class of 4-metrics rather than a specific example. Matter is absent and the coordinate system is time-orthogonal. Allowing for differences in choice of units, his constitutive relations are our (15) with $\varepsilon = \mu = 1$, $g_n = 0$.

The account of Landau and Lifshitz (1962) is similar although differing in two respects: the coordinates are not time-orthogonal, and the 3-metric is chosen according to our (14). Again allowing for different units, putting $\varepsilon = \mu = 1$ and retaining the terms in g_n, their constitutive relations are our (15).

Tanaka (1978) studies electromagnetic wave propagation in a dielectric at rest in the coordinate system of an observer having constant proper acceleration. Consequently his metric is our (16) with zero angular velocity; then (17) gives

$$\boldsymbol{D} = (c/c_1)\varepsilon\varepsilon_0\boldsymbol{E} \qquad \boldsymbol{B} = (c/c_1)\mu\mu_0\boldsymbol{H}$$

and these are the relations used by Tanaka.

From these relations and Maxwell's equations it is straightforward to deduce the vector wave equation

$$\nabla^2\boldsymbol{E} - \left(\frac{n}{c_1}\right)^2\frac{\partial^2\boldsymbol{E}}{\partial T^2} - \frac{1}{cc_1}[\boldsymbol{f}\cdot(\nabla\boldsymbol{E} - \boldsymbol{Ef}/cc_1) + 2\boldsymbol{f}\times(\nabla\times\boldsymbol{E})] = 0 \tag{27}$$

which yields as a special case the scalar equation (31) of Tanaka's paper, after obvious misprints have been corrected. The nominal index of refraction is $n = (\varepsilon\mu)^{1/2}$.

In fact (27) has a wider range of application than might be thought from its derivation which assumed constant acceleration. For, in most applications the time scale of the observer's kinematics is much slower than that of the electromagnetic wave. Consequently the acceleration remains virtually constant during many periods of the wave.

From the structure of (27), without seeking an explicit solution, we can deduce that it describes light travelling in circular arcs with speed c_1/n. To see this, notice that the presence of matter affects only the second term. Suppose we know a solution of (27) for free space. In that solution replace T by T/n; the new solution will satisfy (27) for material of refractive index n. But it follows directly from the free-space metric (Scorgie 1989) that light travels in circular arcs with speed c_1; hence the deduction is established. Moreover, having established the simple time-scaling between the behaviour of light in matter and in free space, we can draw on all the results of the earlier free-space study without further recourse to (27).

Finally, reference may be made to the work of Anderson and Ryon (1969) who give constitutive relations for uniform rotation and constant linear acceleration. Case III in table IV of their paper gives the relations for uniform rotation and their equations

(54) make clear that they are using the 'rotating axes' metric as encountered in the discussion of Heer (1964). With rearrangement of the order of terms to aid comparison, the results in table IV are, for the example of the magnetic relation,

$$(r\mathbf{S})^{-1}\cdot\boldsymbol{B}+r\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{B})=\mu\boldsymbol{H}+r\boldsymbol{v}\times\boldsymbol{E} \tag{28}$$

where as our results are

$$\boldsymbol{B}+\boldsymbol{h}\times(\boldsymbol{h}\times\boldsymbol{B})=\mu\mu_0\boldsymbol{H}+\boldsymbol{h}\times\boldsymbol{E}|c. \tag{29}$$

The apparent discrepancies arise from notational differences. Thus Anderson and Ryon put μ_0 and c equal to unity, and the other differences arise from the fact that they are working in cylindrical polar coordinates: $\mathbf{S}^{-1}$ is just the metric tensor of these coordinates.

Case III in table I of Anderson and Ryon (1969) gives their results for constant linear acceleration, and agreement might be expected with our results and those of Tanaka (1978). But the Anderson and Ryon result is

$$\boldsymbol{B}=\mu\boldsymbol{H}+\boldsymbol{v}\times\boldsymbol{E}-\boldsymbol{v}\times(\boldsymbol{v}\times\boldsymbol{B}) \tag{30}$$

whereas our result, and that of Tanaka, is

$$\boldsymbol{B}=(c/c_1)\mu\mu_0\boldsymbol{H}. \tag{31}$$

There is a real difference here because (30) and (31) apply to different physical situations. Recall that throughout this paper the material has been at rest in the non-inertial coordinate system. Thus for rectilinear motion the material in a plane perpendicular to the direction of motion has a common proper acceleration which differs from plane to plane so that the material remains at rest in the non-inertial coordinate system. Anderson and Ryon, however, assume that the observer and all material particles share a common proper acceleration. This means that planes (of material) perpendicular to the direction of motion do not maintain a constant coordinate distance apart. In short, the material is in motion in the non-inertial coordinate system although, of course, all the particles have a common velocity measured in a fixed inertial system. This is reflected in the fact that, although the field vectors $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ in (30) are measured in the non-inertial frame, $\boldsymbol{v}$ is the common velocity measured in the fixed inertial frame. Thus (31) applies to a solid material whereas (30) applies to material which, in the non-inertial system, is flowing in a way that can be calculated and found to be quite complicated. The essence of the distinction is captured in the old conundrum about the two identical rockets in line ahead configuration and connected by a fine thread: does the thread break?

5. Geometrical optics

Clearly an attempt to study wave propagation using the full constitutive relations (17) is likely to result in a mass of fine detail appropriate only to a specific application. For a broader view it is more profitable to focus attention on the short-wavelength limit contemplated in geometrical optics, acknowledging also the smallness of the non-inertial influences. The detail is straightforward and it will suffice to state the principles governing the approximations that are made.

The numbers fr/c^2 and $\Omega r/c$ are small; hence their product and higher powers of either are nelgected. The wavefunctions contain the factor $\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$, the wavevector

$\boldsymbol{k}$ having a magnitude which varies in inverse proportion to the wavelength; hence the spatial derivative operator ∇ introduces a scalar factor that tends to infinity as the wavelength tends to zero. For example, $|\boldsymbol{r}\times(\nabla\times\boldsymbol{E})|\gg|\boldsymbol{E}|$. And for the reason given in discussing the range of application of (27), time derivatives of the observer's acceleration and angular velocity are neglected.

The approximate vector wave equation that results is

$$\nabla^2\boldsymbol{E}-\left(\frac{n}{c_1}\right)^2\frac{\partial^2\boldsymbol{E}}{\partial T^2}+\frac{2}{c}\frac{\partial}{\partial T}(\boldsymbol{h}\cdot\nabla\boldsymbol{E})-\frac{1}{c^2}[\boldsymbol{f}\cdot\nabla\boldsymbol{E}+2\boldsymbol{f}\times(\nabla\times\boldsymbol{E})]=0 \tag{32}$$

the nominal index of refraction being $n=(\varepsilon\mu)^{1/2}$. The equation in this form resembles (27), remembering that higher-order terms have been dropped and first-order effects of angular velocity have been added. But for further development it is preferable to group together terms involving non-inertial effects, obtaining

$$\nabla^2\boldsymbol{E}-\left(\frac{n}{c}\right)^2\frac{\partial^2\boldsymbol{E}}{\partial T^2}+\frac{2}{c}\frac{\partial}{\partial T}\left(\boldsymbol{h}\cdot\nabla\boldsymbol{E}+\frac{n^2}{c^3}\boldsymbol{r}\cdot\boldsymbol{f}\frac{\partial\boldsymbol{E}}{\partial T}\right)=0. \tag{33}$$

The term in square brackets in (32) has been dropped because in the short-wavelength limit it is negligible compared with the terms retained in (33).

Now write

$$\boldsymbol{E}=\boldsymbol{F}\exp[\mathrm{i}(\varphi-\omega T)]$$

the vector amplitude $\boldsymbol{F}$ and the eikonal φ being functions of position. With nelgect of $\nabla^2\boldsymbol{F}$ due to being small in comparison with the terms retained, we have

$$(\nabla\varphi)^2=(n\omega/c)^2+2(\omega/c)[\boldsymbol{h}\cdot\nabla\varphi-(\omega n^2/c^3)\boldsymbol{r}\cdot\boldsymbol{f}] \tag{34}$$

$$(\nabla^2\varphi)\boldsymbol{F}+2[\nabla\varphi-(\omega/c)\boldsymbol{h}]\cdot\nabla\boldsymbol{F}=0. \tag{35}$$

When (34) has been solved for the eikonal the behaviour of the vector amplitude can be found from (35). The latter aspect is of secondary interest and we confine attention to (34), noting that the factor in square brackets arises from non-inertial effects and is small. If it were zero the plane-wave solution of (34) would be

$$\varphi=(n\omega/c)\boldsymbol{j}\cdot\boldsymbol{r}$$

the unit vector in the direction of travel being $\boldsymbol{j}$.

Accordingly, write

$$\varphi=(n\omega/c)\boldsymbol{j}\cdot\boldsymbol{r}+\theta$$

where θ is a small addition for non-inertial effects. Neglecting $(\nabla\theta)^2$, we find

$$\boldsymbol{j}\cdot\nabla\theta=(\omega/c)(\boldsymbol{h}\cdot\boldsymbol{j}-n\boldsymbol{r}\cdot\boldsymbol{f}/c^2)$$

which is satisfied by

$$\nabla\theta=(\omega/c)[\boldsymbol{h}-(n\boldsymbol{r}\cdot\boldsymbol{f}/c^2)\boldsymbol{j}] \qquad \nabla\varphi=(\omega/c)[n\boldsymbol{j}+\boldsymbol{h}-(n\boldsymbol{r}\cdot\boldsymbol{f}/c^2)\boldsymbol{j}].$$

The optical path length of a ray from P_1 to P_2 is

$$\Delta=\int_1^2\partial\boldsymbol{r}\cdot\nabla\varphi$$

the integral being taken along the ray. Thus

$$\Delta = \Delta(\sigma) + \Delta(\boldsymbol{\Omega}) + \Delta(\boldsymbol{f})$$

$$\Delta(\sigma) = n\omega\sigma/c \qquad \sigma = \text{coordinate length of ray} \tag{36}$$

$$\Delta(\boldsymbol{\Omega}) = (\omega/c^2)\boldsymbol{\Omega}\cdot\int_1^2 \boldsymbol{r}\times\partial\boldsymbol{r} \tag{37}$$

$$\Delta(\boldsymbol{f}) = -(n\omega\sigma/c^3)\boldsymbol{\rho}\cdot\boldsymbol{f} \tag{38}$$

the position vector of the centroid of the ray being $\boldsymbol{\rho}$.

The last three equations are the principal results of the short-wavelength approximation; they show that the optical path length consists of three additive components. The first, given by (36), greatly exceeds the other two and exists in an inertial system. The small contributions arising from the angular velocity and the translational acceleration of the observer are given by (37) and (38), respectively. Only the contribution from angular velocity does not depend on the nominal refractive index of the matter; and only that contribution changes sign with reversal of the direction of travel of the ray.

These relations are immediately applicable to an understanding of the Sagnac effect that underlies the principle of the optical gyroscope. For a ray travelling in a closed circuit, (37) gives

$$\Delta(\boldsymbol{\Omega}) = 2(\omega/c^2)\boldsymbol{\Omega}\cdot\boldsymbol{A}$$

the vector area enclosed by the circuit being

$$\boldsymbol{A} = \tfrac{1}{2}\int \boldsymbol{r}\times\partial\boldsymbol{r}.$$

If the circuit is traversed in the opposite direction the vector area changes sign; hence the difference between the two values of optical path length is

$$\delta = 4(\omega/c^2)\boldsymbol{\Omega}\cdot\boldsymbol{A}. \tag{39}$$

The other two contributions to the optical path length remain unaltered by reversal of direction of travel round the circuit and hence cancel when the difference is formed. The upshot is that, if two rays travel in opposite directions round the closed circuit and are reunited at the launch point, the difference between their optical path lengths is given by (39). This difference can be detected and used to measure the angular velocity.

Several points may be noted. In accordance with an earlier remark, (39) does not depend on the nominal refractive index of the material traversed by the rays. And the observer's acceleration is also absent. This feature is worth remarking because most treatments of the Sagnac effect simply ignore acceleration rather than present arguments justifying its neglect. And of course it has to be remembered that the present treatment is approximate; a more accurate account would no doubt disclose some dependence on acceleration, though this is probably well below the level of experimental detection. Furthermore, although (39) or its equivalent appears in all accounts of the Sagnac effect, there may be subtle differences in the meanings of the symbols. Thus the angular velocity is here defined in terms of the geometry of the worldline of the point where the two oppositely travelling rays are launched and subsequently reunited. Likewise, the vector area is measured in the non-inertial coordinate system. For example, if the closed circuit is, when represented in the inertial frame of its centre, a circle rotating about an axis through the centre and perpendicular to the plane of the circle, it is not difficult to see that the circuit becomes an ellipse in the non-inertial frame of an observer attached to a point on its circumference. But here again such subtleties would be significant only at better levels of approximation than are likely to be justified experimentally.

In view of the practicability of the optical gyroscope, it is natural to speculate on the possibility of devising an optical accelerometer that would be based on (38) just as the gyroscope is based on (37). But it is easy to see that the hope is indeed forlorn because the contribution from (38) is completely swamped by that from (36); and, crucially, the latter cannot be made to cancel as in the gyroscope. Of course, ideally, cancellation is possible by arranging two rays at right angles to each other; but this would require equality of physical length to an impracticably high degree of accuracy.

Finally, it would be desirable to compare (36), (37) and (38) with the results of other authors. However, it has not been possible to find other treatments that quote a complete set of comparable results. (Indeed this paper owes its existence in some measure to that fact.) Other treatments ignore the effect of translational acceleration when accompanied by rotation; hence there is nothing with which (38) may be compared. On the other hand, (36) is perhaps trivially obvious. This leaves (37), and here we may mention two sources, namely Chow *et al* (1985) and Post (1967).

As remarked previously, the interesting feature of (37) is its independence of the refractive index of the comoving material traversed by the light ray. At first glance equation (1.24) of Chow *et al* (1985) seems to confirm our (37), allowing for the fact that their (1.24) is restricted to a closed path, whereas our (37) applies to any path. However, closer examination reveals that the treatment of Chow *et al* is throughout restricted to free space; hence the absence of an index of refraction from their results cannot serve to confirm our conclusion in this respect.

Post (1967) gives two treatments which he calls, respectively, geometric optical theory and physical optical theory. Dealing first with the geometric optical theory, his equation (48)

$$\Delta\tau = \frac{2}{c^2}\int n^2(1-\alpha)\boldsymbol{v}\cdot d\boldsymbol{r}$$

is that one of his results that seems to be most nearly comparable. This equation gives the time difference between two rays propagating in opposite directions round a closed path subject to the velocity field $\boldsymbol{v}$. The important feature is the occurrence of n, the refractive index of the comoving material, in association with α which he describes as 'a coefficient of drag similar to but not necessarily identical with the Fresnel–Fizeau coefficient of drag for translational motion'. At his equation (2) Post indicates that, for a non-dispersive medium in translational motion, he takes $\alpha = 1 - n^{-2}$. Of course this value would render his equation (48) quoted above independent of the refractive index. However, his geometric optical theory is silent on the value of α and it is necessary to turn to his physical optical theory for enlightenment. His conclusion from that theory is that, for rotational motion, the relation $\alpha = 1 - n^{-2}$ is indeed satisfied, but only if the relative permeability of the comoving material is unity.

It is difficult to be certain, but this conclusion seems to stem ultimately from Post's handling of the constitutive relations. He splits the constitutive tensor into two parts associated, respectively, with free space and matter, and then transforms the two parts independently. This seems a very dubious procedure; indeed Anderson and Ryon (1969) claim that it leads to inconsistencies.

Certainly in his review of the experimental evidence, Post (1967) concludes that it 'demonstrates beyond doubt' that the fringe shift observed in the Sagnac effect 'does not depend on a comoving refracting medium in the path of the beam'. However, it is likely that the experiments used material whose relative permeability differed little from unity; consequently this evidence is not conclusive.

6. Discussion

In his treatment of Maxwell's equations in matter that is moving uniformly, Synge (1965) sets aside what he describes as 'the far more difficult problem of electromagnetism in a body in accelerated motion, e.g. in rotation'. The problem has two aspects: the physics and the formalism. The physics has been avoided in this article by the assumption that, to a momentarily comoving inertial observer, the constitutive relations are unaffected by the acceleration of the material. A defence can be based on two grounds: the assumption is likely to be approximately true of all materials at low acceleration, whereas any other is likely to be restricted to specific materials; and it provides a benchmark against which others can be viewed.

This article has been directed solely at the second aspect: given the above assumption, to produce a formulation in terms of the kinematics of the observer's worldline. The central result is the constitutive relations (15), (17) and their approximate form (18). These relations are to be coupled with the universal form of the field equations (2).

The universality of form of the field equations has the consequence that most of the variety, both of material properties and of observer's motion, is carried by the constitutive relations. This separation between the field equations and the constitutive relations results from the use of four distinct vectors $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{D}$ to describe the field even in free space. It may be remarked that even some of the authors who define the free space field in inertial coordinates by $\boldsymbol{E}$ and $\boldsymbol{B}$ alone find it useful to introduce $\boldsymbol{D}$ and $\boldsymbol{H}$ also when non-inertial coordinates are encountered. This is surely an argument for always using the four distinct vectors.

Although the field equations (2) retain their form in all circumstances, their content depends to some extent on individual circumstances. In particular, this arises because the determinant of the metric of the 3-space enters. And if the coordinate system is not time-orthogonal there is a choice of 3-metric: it may be the space part of the 4-metric or it may be as defined by (14). The latter has two points in its favour. Firstly, as has been shown, the constitutive relations are slightly simplified. Secondly, there is a sense in which the 3-metric (14) provides the true measure of distance in the 3-space.

The existence of two conceivable 3-metrics might seem to be something of an embarrassment: how should the choice be determined? But in fact the difference between the two, being second order in $\Omega r/c$, is generally not detectable experimentally. Of course on another view of this situation it is a matter for regret that potential variety in the fine detail has apparently not been realized because measurements are not sufficiently accurate. Something of the same thought attaches to the perennial question: what is the 'correct' metric (if, indeed, one exists) when rotation is encountered? The point has arisen in discussing the work of Heer (1964), and a distinction has been drawn between two situations. In one, 'transformation to rotating axes', the origin of coordinates is at rest in an inertial frame but the space axes rotate. In the other the origin of coordinates moves in a circle and so suffers centripetal acceleration. Heer uses the metric appropriate to the first although his problem requires the second. But again the difference between the two metrics is second order and hence insignificant experimentally.

The metric (16) for an observer in completely general motion has proved useful but is of wider interest in its own right since it is the basis for all physical measurements that he makes. The parameters describing the kinematics of the observer's worldline are the three curvatures. The first curvature is a measure of the observer's proper acceleration f which appears in the metric through c_1. The second and third curvatures are proportional to components of the angular velocity $\boldsymbol{\Omega}$ of the space axes which appears in the metric through the vector $\boldsymbol{h}$. Thus, given the observer's worldline, it is straightforward to apply the Frenet equations to calculate f and $\boldsymbol{\Omega}$ and so arrive at the metric (16). Its form shows why, in general, physics in non-inertial coordinates is likely to be more sensitive to rotation than to acceleration. For, rotation produces the term in $\boldsymbol{h}$ for which there is no counterpart in an inertial frame. In contrast, acceleration merely effects a slight change in a term that already exists in an inertial frame: the inertial speed of light c is replaced by c_1.

The point is illustrated by the short-wavelength limit of the wave equation. Acceleration does contribute to the optical length of a light ray in the non-inertial system, but its contribution is likely to be insignificant even compared with the small contribution from angular velocity. By far the greatest part of the optical path length is the inertial contribution. The optical gyroscope is feasible because both oppositely travelling light rays are equally affected by the inertial contribution which consequently disappears from the difference between their optical path lengths. Inability to arrange such cancellation in the optical accelerometer accounts for its impracticability.

Appendix. The intrinsic coordinate system

The observer's proper time being T, differentiation with respect to cT is denoted by a superior dot. With $\boldsymbol{X}_4$ denoting unit tangent to the observer's worldline, write $\dot{\boldsymbol{X}}_4 = K_1\boldsymbol{X}_1$, where $\boldsymbol{X}_1$ is a unit spacelike vector and K_1 is a scalar. Write $\dot{\boldsymbol{X}}_1 = p\boldsymbol{X}_4 + K_2\boldsymbol{X}_2$ where p and K_2 are scalars and $\boldsymbol{X}_2$ is a unit spacelike vector orthogonal to $\boldsymbol{X}_1$. Then $p = -\boldsymbol{X}_4 \cdot \dot{\boldsymbol{X}}_1 = K_1$. Continuing in this fashion leads to the Frenet equations in spacetime

$$\begin{aligned} &\dot{\boldsymbol{X}}_4 = K_1\boldsymbol{X}_1 \qquad &&\dot{\boldsymbol{X}}_1 = K_1\boldsymbol{X}_4 + K_2\boldsymbol{X}_2 \\ &\dot{\boldsymbol{X}}_2 = -K_2\boldsymbol{X}_1 + K_3\boldsymbol{X}_3 \qquad &&\dot{\boldsymbol{X}}_3 = -K_3\boldsymbol{X}_2. \end{aligned} \tag{A.1}$$

Because the observer's 4-velocity is $c\boldsymbol{X}_4$ his 4-acceleration is

$$\boldsymbol{f} = f\boldsymbol{X}_1 \qquad f = c^2K_1.$$

In the observer's 3-space it is convenient to use ordinary vector notation in which the dot and cross respectively denote scalar and vector multiplication. Define the angular velocity

$$\boldsymbol{\Omega} = c(K_3\boldsymbol{X}_1 + K_2\boldsymbol{X}_3).$$

Then (A.1) can be written

$$\begin{aligned} &\mathrm{d}\boldsymbol{X}_4/\mathrm{d}\tau = (f/c)\boldsymbol{X}_1 \qquad &&\mathrm{d}\boldsymbol{X}_1/\mathrm{d}T = \boldsymbol{\Omega}\times\boldsymbol{X}_1 + (f/c)\boldsymbol{X}_4 \\ &\mathrm{d}\boldsymbol{X}_2/\mathrm{d}T = \boldsymbol{\Omega}\times\boldsymbol{X}_2 \qquad &&\mathrm{d}\boldsymbol{X}_3/\mathrm{d}T = \boldsymbol{\Omega}\times\boldsymbol{X}_3. \end{aligned}$$

Thus $\boldsymbol{\Omega}$ is the angular velocity of the space axes.

The position vector of an event P in the observer's 3-space is $\boldsymbol{r}=x^m\boldsymbol{X}_m$ and its spacetime coordinates are x^α with $x^4=cT$. Since spacetime is flat, an ordered pair of events defines a 4-vector. To find the metric in this coordinate system let O be an arbitrary spacetime origin and let Q be the event occupied by the observer. Let the spacetime vectors be

$$OQ=\boldsymbol{W} \qquad QP=\boldsymbol{Y}=x^m\boldsymbol{X}_m \qquad OP=\boldsymbol{R}=\boldsymbol{W}+\boldsymbol{Y}.$$

The 4-velocity of a material particle at P having proper time τ is

$$V=\mathrm{d}\boldsymbol{R}/\mathrm{d}\tau=c\beta(\dot{\boldsymbol{W}}+\dot{\boldsymbol{Y}}) \qquad \beta=\mathrm{d}T/\mathrm{d}\tau.$$

Also

$$\dot{\boldsymbol{W}}=\boldsymbol{X}_4 \qquad \dot{\boldsymbol{Y}}=\dot{x}^m\boldsymbol{X}_m+x^m\dot{\boldsymbol{X}}_m.$$

The position vector being $x^m\boldsymbol{X}_m=\boldsymbol{r}$, the coordinate velocity is

$$c\dot{x}^m\boldsymbol{X}_m=\boldsymbol{u}=\partial\boldsymbol{r}/\mathrm{d}T$$

the operator $\partial/\mathrm{d}T$ acting on vector components but treating the basis vectors as constant. Then substitution for $\dot{\boldsymbol{X}}_m$ from the Frenet relations gives

$$\boldsymbol{V}=\beta(\boldsymbol{u}+\boldsymbol{\Omega}\times\boldsymbol{r}+c_1\boldsymbol{X}_4) \qquad c_1=(1+\boldsymbol{r}\cdot\boldsymbol{f}/c^2)c. \tag{A.2}$$

Since the square of a 4-velocity is $-c^2$, we have

$$\beta=(c/c_1)\{1-[u^2+(\boldsymbol{\Omega}\times\boldsymbol{r})^2+2\boldsymbol{u}\cdot\boldsymbol{\Omega}\times\boldsymbol{r}]c_1^{-2}\}^{-1/2}.$$

Defining

$$\boldsymbol{h}=\boldsymbol{\Omega}\times\boldsymbol{r}/c$$

and recalling that the square of the element of interval at P is $\mathrm{d}s^2=-c^2\,\mathrm{d}\tau^2$, we find

$$\mathrm{d}s^2=\partial\boldsymbol{r}\cdot\partial\boldsymbol{r}+2\boldsymbol{h}\cdot\partial\boldsymbol{r}\,\mathrm{d}x^4-[(c_1/c)^2-h^2](\mathrm{d}x^4)^2.$$

From its construction, the vector set $\boldsymbol{X}_\alpha$ is an orthonormal basis for describing the geometry of events from the point of view of the non-inertial observer. But in general it will not be a coordinate basis: there will not exist coordinates θ^α such that $\boldsymbol{X}_\alpha=\partial/\partial\theta^\alpha$. On the other hand, the coordinate basis $\boldsymbol{F}_\alpha=\partial/\partial x^\alpha$ is in general not orthonormal.

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